Linear Algebra II

21/03/2013, Thursday, 14:00-16:00

1 (7+18=25 pts) **Inner product spaces and Gram-Schmidt process**

Let V be an inner product space and let $\|v\| = \langle v, v\rangle^{\frac{1}{2}}$.

(a) Show that $u = v$ if and only if $\langle u, v\rangle = \|u\|^2 = \|v\|^2$.

(b) Suppose that the vectors x, y, z form a basis for a subspace $S \subseteq V$ and satisfy

$$\begin{array}{lll} \langle x, x\rangle = 1, & \langle x, y\rangle = 1, & \langle x, z\rangle = 0, \\ & \langle y, y\rangle = 2, & \langle y, z\rangle = 1, \\ & & \langle z, z\rangle = 2. \end{array}$$

Apply the Gram-Schmidt process and obtain an orthonormal basis for the subspace S.

Required Knowledge: **inner product, Gram-Schmidt process.**

Solution:

(1a):

"only if": Clearly, $\langle u, v\rangle = \|u\|^2 = \|v\|^2$ whenever $u = v$.

"if": Suppose that $\langle u, v\rangle = \|u\|^2 = \|v\|^2$. Note that

$$\begin{aligned} \|u - v\|^2 &= \langle u - v, u - v\rangle \\ &= \langle u, u\rangle - \langle u, v\rangle - \langle v, u\rangle + \langle v, v\rangle \\ &= 0 \quad (\text{since } \langle u, v\rangle = \|u\|^2 = \|v\|^2). \end{aligned}$$

Therefore, we can conclude that $u = v$.

(1b): By applying the Gram-Schmidt process, we obtain:

$$
\begin{aligned}
u_1 &= \frac{x}{\|x\|} \\
u_1 &= x \\
u_2 &= \frac{y - p_1}{\|y - p_1\|} & p_1 &= \langle y, x\rangle x = x \\
& & y - p_1 &= y - x \\
& & \|y - p_1\|^2 &= \langle y - x, y - x\rangle \\
& & &= \langle y, y\rangle - \langle y, x\rangle - \langle x, y\rangle + \langle x, x\rangle \\
& & &= 2 - 1 - 1 + 1 \\
& & \|y - p_1\| &= 1 \\
u_2 &= y - x \\
u_3 &= \frac{z - p_2}{\|z - p_2\|} & p_2 &= \langle z, x\rangle x + \langle z, y - x\rangle (y - x) = y - x \\
& & z - p_2 &= z - y + x \\
& & \|z - p_2\|^2 &= \langle z - y + x, z - y + x\rangle \\
& & &= \langle z, z\rangle + \langle y, y\rangle + \langle x, x\rangle - 2\langle z, y\rangle + 2\langle z, x\rangle - 2\langle y, x\rangle \\
& & &= 2 + 2 + 1 - 2 + 0 - 2 \\
& & &= 1 \\
& & \|z - p_2\| &= 1 \\
u_3 &= z - y + x.
\end{aligned}
$$

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2 (10+10 = 20 pts) Eigenvalues and diagonalization

Let $M \in \mathbb{R}^{n\times n}$ be a normal matrix.

(a) Show that M is symmetric if all its eigenvalues are real.

(b) Show that if $a + ib$ ($a, b \in \mathbb{R}$) is an eigenvalue of M then $\sqrt{a^2 + b^2}$ is a singular value of M.

REQUIRED KNOWLEDGE: **eigenvalues, eigenvectors, normal matrices, unitary matrices, diagonalization by unitary matrices, singular values.**

SOLUTION:

(2a):

Since M is normal, there exist a unitary matrix U and a diagonal matrix D such that $M = UDU^H$. Also, we know that the diagonal elements of D much be the eigenvalues of M and the columns of U must be eigenvectors. Since all eigenvalues are real, we have $D^H = D^T = D$ and also $U^H = U^T$ as all eigenvectors can be chosen to be real-valued vectors. Then, we have

$$
\begin{aligned}
M &= UDU^H = UDU^T \\
M^H &= M^T = UD^HU^H = UDU^T.
\end{aligned}
$$

Therefore, M is symmetric.

(2b):

Since M is normal, there exist a unitary matrix U and a diagonal matrix D such that $M = UDU^H$. Then, we have

$$\begin{aligned} M^T M &= M^H M && (\text{since } M \in \mathbb{R}^{n\times n}) \\ &= UD^H U^H U D U^H \\ &= UD^H D U^H && (\text{since } U \text{ is unitary}) \end{aligned}$$

Note that the diagonal elements of D are eigenvalues of M and those of $D^H D$ are eigenvalues of $M^T M$. Hence, we can conclude that if $a + ib$ is an eigenvalue of M then $(a+ib)(a-ib) = a^2+b^2$ must be an eigenvalue of $M^T M$. Since singular values are square roots of the eigenvalues of $M^T M$, $\sqrt{a^2+b^2}$ must be a singular value of M.

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3 (10+10 = 20 pts) **Positive definiteness**

Consider the matrix

$$M = \begin{bmatrix} a & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & a \end{bmatrix}$$

where a is real number. For which values of a

(a) is M positive definite?

(b) is M negative definite?

REQUIRED KNOWLEDGE: **positive/definite matrices, leading principal minor test for positive definiteness.**

SOLUTION:

(3a):

A symmetric matrix is positive definite if and only if all its leading principal minors are positive. For the matrix we have, the leading principal minors can be computed as follows:

$$\det([a]) = a$$

$$\det\left(\begin{bmatrix} a & 1 \\ 1 & 1 \end{bmatrix}\right) = a - 1$$

$$\det\left(\begin{bmatrix} a & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & a \end{bmatrix}\right) = a^2 + 1 + 1 - a - 1 - a = a^2 - 2a + 1.$$

Therefore, M is positive definite if and only if

$$\begin{aligned} a &> 0 \\ a - 1 &> 0 \\ (a-1)^2 &> 0. \end{aligned}$$

These inequalities are satisfied if and only if $a > 1$.

(3b):

Note that M is negative definite if and only if $-M$ is positive definite. Then, we can immediately conclude that M is never negative definite as it has a positive diagonal element. Alternatively, one can again employ the leading principal minor test:

$$\det([-a]) = -a$$

$$\det\left(\begin{bmatrix} -a & 1 \\ 1 & -1 \end{bmatrix}\right) = a - 1$$

$$\det\left(\begin{bmatrix} -a & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -a \end{bmatrix}\right) = -a^2 - 1 - 1 + a + 1 + a = -(a^2 - 2a + 1).$$

Hence, we can conclude that M is negative definite if and only if the inequalities

$$a > 0$$
$$a - 1 > 0$$
$$-(a - 1)^2 > 0$$

are feasible. However, the last inequality is never satisfied since a is a real number. Therefore, M cannot be negative definite.

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4 (15+10=25 pts) **Singular value decomposition**

Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(a) Find a singular value decomposition of A.

(b) Find a matrix B having the singular values $\sqrt{3}$, $\sqrt{2}$, and $\frac{\sqrt{6}}{2}$ and satisfying $\|A - B\|_F = \frac{\sqrt{6}}{2}$.

Required Knowledge: **singular value decomposition, lower rank approximations.**

Solution:

(4a):

Note that

$$A^T A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $A^T A$ is diagonal, we can take $V = I$. Also, we have $\sigma_1 = \sqrt{3}$, $\sigma_2 = \sqrt{2}$, and $\sigma_3 = 0$. Since the number of non-zero singular values is 2, we have $\operatorname{rank}(A) = 2$. Then, we get

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

To obtain the last column of U, we need to find an orthonormal basis for the null space of A^T. Note that

$$0 \quad A^T x \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \begin{bmatrix} x_1 \quad x_2 + x_3 \\ x_1 + x_2 \\ 0 \end{bmatrix}.$$

Therefore, we get

$$\mathcal{N}(A^T) \quad \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right).$$

This leads to

$$u_3 \quad \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Consequently, a singular decomposition for A can be given as

$$A \quad \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(4b):

Note that $\sqrt{2} > \sqrt{6}/2$. It follows from best lower rank approximations by singular value decomposition that the matrix

$$B \quad \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{6}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

satisfies $\|A \quad B\|_F \quad \sqrt{6}/2$. Therefore, we can choose

$$B \quad \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{6}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1/2 \\ 1 & 1 & 1/2 \\ 1 & 0 & 1 \end{bmatrix}.$$